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THE INFLUENCE OF DISSIPATIVE AND CONSTANT FORCES ON THE FORM AND STABILITY OF STEADY MOTIONS OF MECHANICAL SYSTEMS WITH CYCLIC COORDINATES†

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Mechanical systems with cyclic coordinates subject to dissipative forces with complete dissipation and constant forces applied only to the cyclic variables are considered. Problems of the existence of steady motions in such systems and the conditions for their stability are discussed. It is shown, in particular, that if the Rayleigh function is proportional to the kinetic energy, the stability conditions for the steady motions of the system are the same as or (under certain assumptions) similar to such conditions for steady motions of a corresponding conservative system. The example of a physical pendulum is used to show that such conclusions are generally false: dissipative and constant forces may cause destabilization of stable motions of the system. © 1998 Elsevier Science Ltd. All rights reserved.

1. Consider a conservative mechanical system with n degrees of freedom, assuming that neither the kinetic energy T nor the potential energy V of the system depend on $m < n$ generalized coordinates. Denote these coordinates by $\mathbf{s} = (s_1, \ldots, s_m)^{\mathrm{T}}$ and the others by $\mathbf{r} = (r_1, \ldots, r_k)^{\mathrm{T}}$ ($k + m = n$; the superscript T denotes transposition). The coordinates $\mathbf{r}$ and $\mathbf{s}$, as is well known, are referred to as positional and cyclic coordinates respectively. Thus

$$T = \tfrac{1}{2}[(\mathbf{A}(\mathbf{r})\dot{\mathbf{r}},\dot{\mathbf{r}}) + 2(\mathbf{B}(\mathbf{r})\dot{\mathbf{r}},\dot{\mathbf{s}}) + (\mathbf{C}(\mathbf{r})\dot{\mathbf{s}},\dot{\mathbf{s}})],\quad V = V(\mathbf{r})$$

where $\mathbf{A}$ and $\mathbf{C}$ are symmetric $k \times k$ and $m \times m$ matrices, respectively and $\mathbf{B}$ is an $m \times k$ matrix, such that

$$\begin{Vmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{C} \end{Vmatrix}$$

is the matrix of a positive-definite quadratic form.

Two formulations of the problem of steady motion are widely used for systems with cyclic coordinates. In one case, it is assumed that no additional forces are acting on the system. Here, the system admits of m cyclic integrals $\partial T/\partial \mathbf{s} = \mathbf{c}$ and may perform steady motions of the form

$$\mathbf{r} = \mathbf{r}_c^0,\quad \dot{\mathbf{r}} = \mathbf{0},\quad \mathbf{s} = \dot{\mathbf{s}}_c^0 t + \mathbf{s}^0,\quad \dot{\mathbf{s}} = \dot{\mathbf{s}}_c^0 = \mathbf{C}^{-1}(\mathbf{r}_c^0)\mathbf{c} \tag{1.1}$$

where the constants $\mathbf{s}^0$ are arbitrary and the constants $\mathbf{r}_C^0$ are determined from the system

$$\partial V_c / \partial \mathbf{r} = 0,\quad V_c = V(\mathbf{r}) + \tfrac{1}{2}(\mathbf{C}^{-1}(\mathbf{r})\mathbf{c},\mathbf{c}) \tag{1.2}$$

In the other case, it is assumed that the system is subject to control forces that keep the generalized cyclic velocities constant in all motions: $\dot{\mathbf{s}} \equiv \boldsymbol{\omega}$. In that case the system admits of relative equilibria of the form

$$\mathbf{r} = \mathbf{r}_\omega^0,\quad \dot{\mathbf{r}} = \mathbf{0},\quad \mathbf{s} = \boldsymbol{\omega} t + \mathbf{s}^0,\quad \dot{\mathbf{s}} = \boldsymbol{\omega} \tag{1.3}$$

where the constants $\mathbf{s}^0$ are arbitrary and the constants $\mathbf{r}_\omega^0$ are determined from the system

$$\partial V_\omega / \partial \mathbf{r} = \mathbf{0},\quad V_\omega = V(\mathbf{r}) - \tfrac{1}{2}(\mathbf{C}(\mathbf{r})\boldsymbol{\omega},\boldsymbol{\omega}) \tag{1.4}$$

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The first form of the problem goes back to Routh [1], and the second, to Poincaré [2]. By Lagrange's theorem, the relative equilibrium (1.3) is stable (with respect to $\mathbf{r}$ and $\dot{\mathbf{r}}$) if the changed potential V_ω takes a strictly minimum value at the point $\mathbf{r}_\omega^0$ (see [3–6]). By Routh's theorem, the steady motion (1.1) is stable (with respect to $\mathbf{r}$, $\dot{\mathbf{r}}$ and $\dot{\mathbf{s}}$) if the reduced potential V_c takes a strictly minimum value at the point $\mathbf{r}_C^0$ (see [1, 7–9]). Note that there is a complete correspondence between the steady motions (1.1) and the relative equilibria (1.3), provided the arbitrary constants $\mathbf{c}$ and ω have a certain relationship; the stability conditions for both settings also correspond in a certain sense (see [2, 10–14]).

However, both settings of the problem are in a sense idealized, since they make no allowance for the effect of dissipative forces, which are always present in real systems and, in particular, destroy the cyclic integrals of the free system. In addition, it is extremely difficult in practice to produce control forces that keep the generalized cyclic velocities constant in all motions of the controlled system (independently of the variation of the positional variables).

In this paper it is assumed that, apart from potential forces, the system is also subject to dissipative forces with complete dissipation, the derivatives of the Rayleigh function $f\Phi$ (f being a positive parameter), where

$$\Phi = \tfrac{1}{2}[(\mathbf{D}(\mathbf{r})\dot{\mathbf{r}},\dot{\mathbf{r}}) + 2(\mathbf{E}(\mathbf{r})\dot{\mathbf{r}},\dot{\mathbf{s}}) + (\mathbf{F}(\mathbf{r})\dot{\mathbf{s}},\dot{\mathbf{s}})]$$

and constant forces $f\mathbf{p}$, where $\mathbf{D}$ and $\mathbf{F}$ are symmetric $k \times k$ and $m \times m$ matrices, respectively, $\mathbf{E}$ is an $m \times k$ matrix and $\mathbf{p}$ is an n-dimensional vector of the form $(0,\dots 0, p_1,\dots,p_m)^{\mathrm{T}}$ (the zero components of $\mathbf{p}$ correspond to the positional variables). Under these conditions, the equations of motion of the system have the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{r}}} = \frac{\partial T}{\partial \mathbf{r}} - \frac{\partial V}{\partial \mathbf{r}} - f\frac{\partial \Phi}{\partial \dot{\mathbf{r}}}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{s}}} = f\mathbf{p} - f\frac{\partial \Phi}{\partial \dot{\mathbf{s}}} \tag{1.5}$$

This formulation of the problem is due to Pozharitskii [15], who assumed that $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{F}$ are constant matrices. Non-constant matrices $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{F}$ have been considered in a few special cases [16]. In both the aforementioned papers, however, it was also assumed that the vector $\mathbf{p}$ has no zero components, i.e. that constant forces were applied with respect to all variables. Note that the actual production of constant forces for positional variables is extremely difficult, whereas that is easily done for cyclic variables in many applied problems (a gyroscope in gimbals, a body with slide-wire drive, etc.).

2. Let us consider the system described by Eqs (1.5), assuming that $\Phi = T$, that is, the Rayleigh dissipation function is proportional to the kinetic energy. Such dissipation is a model of, for example, the influence of a resistant medium. In that case system (1.5) becomes

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{r}}} = \frac{\partial T}{\partial \mathbf{r}} - \frac{\partial V}{\partial \mathbf{r}} - f\frac{\partial T}{\partial \dot{\mathbf{r}}}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{s}}} = f\left(\mathbf{p} - \frac{\partial T}{\partial \dot{\mathbf{s}}}\right) \tag{2.1}$$

System (2.1) admits of particular integrals

$$\partial T/\partial\dot{\mathbf{s}} = \mathbf{p} \tag{2.2}$$

which determine the invariant set of the system. This invariant set is asymptotically stable in the large, since it follows from the second group of system (2.1) that

$$(\partial T/\partial\dot{\mathbf{s}} - \mathbf{p}) = (\partial T/\partial\dot{\mathbf{s}} - \mathbf{p})_0 e^{-f(t-t_0)}$$

(recall that $f > 0$; the subscript zero means that the relevant expression is evaluated at $t = t_0$).

Let us consider the system in the asymptotically stable invariant set (2.2). Solving (2.2) for the generalized cyclic velocities, we obtain

$$\dot{\mathbf{s}} = \mathbf{C}^{-1}(\mathbf{r})(\mathbf{p} - \mathbf{B}(\mathbf{r})\dot{\mathbf{r}}) \tag{2.3}$$

Introducing Routh's function

$$R = [T - V - (\mathbf{p},\dot{\mathbf{s}})]_{(2.3)} = R(\mathbf{r};\dot{\mathbf{r}};\mathbf{p}) = R_2 + R_1 + R_0$$

$$R_2 = \tfrac{1}{2}(\mathbf{M}(\mathbf{r})\dot{\mathbf{r}},\dot{\mathbf{r}}), \quad R_1 = (\mathbf{g}_p(\mathbf{r}),\dot{\mathbf{r}}), \quad R_0 = -V_p(\mathbf{r})$$

$$\mathbf{M}(\mathbf{r}) = \mathbf{A} - \mathbf{B}'\mathbf{C}^{-1}\mathbf{B}, \quad \mathbf{g}_p(\mathbf{r}) = \mathbf{B}'\mathbf{C}^{-1}\mathbf{p}, \quad V_p(\mathbf{r}) = V + \tfrac{1}{2}(\mathbf{C}^{-1}\mathbf{p},\mathbf{p})$$

Thus, the motion of the system in the asymptotically stable invariant set (2.2) is described by Routh's equations

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{\mathbf{r}}}=\frac{\partial R}{\partial \mathbf{r}}-f\frac{\partial R}{\partial \dot{\mathbf{r}}} \tag{2.4}$$

Taking the structure of the Routh function into consideration, we rewrite system (2.4) as

$$\frac{d}{dt}\frac{\partial R_2}{\partial \dot{\mathbf{r}}}=\frac{\partial R_2}{\partial \mathbf{r}}+\mathbf{G}_p\dot{\mathbf{r}}-\frac{\partial V_p}{\partial \mathbf{r}}-f\mathbf{g}_p-f\frac{\partial R_2}{\partial \dot{\mathbf{r}}} \tag{2.5}$$

$$\left(\mathbf{G}_p=\left(\frac{\partial \mathbf{g}_p}{\partial \mathbf{r}}\right)^T-\left(\frac{\partial \mathbf{g}_p}{\partial \mathbf{r}}\right),\quad \text{i.e. } \mathbf{G}_p^T=-\mathbf{G}_p\right)$$

Equations (2.5) describe the motion of a certain system with degrees of freedom in which R_2 plays the part of the kinetic energy. We will refer to this as the "reduced" system. Clearly, the "reduced" system is subject to the action of potential forces—the derivatives of the "reduced" potential V_p, gyroscopic forces $\mathbf{G}_p\dot{\mathbf{r}}$, dissipative forces—the derivatives of the Rayleigh function fR_2 and generalized positional forces $f\mathbf{g}_p$ (which need not be potential forces).

Note that if $f = 0$, the particular integrals (2.2) become general integrals, and the system and the function V_p are identical with the reduced system in Routh's sense and the reduced potential (without quotation marks), respectively.

The equilibrium positions $\mathbf{r} = \mathbf{r}_p^0$, $\dot{\mathbf{r}} = 0$ of the "reduced" system correspond to the steady motions

$$\mathbf{r}=\mathbf{r}_p^0,\quad \dot{\mathbf{r}}=0,\quad \mathbf{s}=\dot{\mathbf{s}}_p^0t+\mathbf{s}^0,\quad \dot{\mathbf{s}}=\dot{\mathbf{s}}_p^0=\mathbf{C}^{-1}(\mathbf{r}_p^0)\mathbf{p} \tag{2.6}$$

of the system we are considering. The constants $\mathbf{s}^0$ are arbitrary and the constants $\mathbf{r}_p^0$ are determined from the system

$$\partial V_p/\partial \mathbf{r}+f\mathbf{g}_p=0 \tag{2.7}$$

It is obvious that system (2.7) becomes (1.2) and the steady motions (2.6) become the steady motions (1.1) not only when $f = 0$ (and $\mathbf{p} = \mathbf{c}$), which is natural, but also when $\mathbf{g}_p \equiv 0$. This happens provided that $\mathbf{B} \equiv 0$, which means that the kinetic energy does not contain products of positional and cyclic velocities. Then Eq. (2.5) admits of a generalized energy equation

$$d(R_2+V_p)/dt=-2fR_2\leqslant 0 \tag{2.8}$$

Thus the following propositions are true (cf. [16]).

Theorem 2.1. If $\mathbf{B} \equiv 0$ and $\mathbf{p} = \mathbf{c}$, then the steady motions of a system subject to dissipative forces—the derivatives of a Rayleigh function proportional to the kinetic energy—and constant forces applied only with respect to the cyclic variables, are identical with the steady motions of the corresponding conservative system.

Theorem 2.2. If $\mathbf{B} \equiv 0$, the "reduced" potential V_p has a strict local minimum at the point $\mathbf{r}_p^0$ and this point is isolated from the other stationary points of the "reduced" potential, then the steady motion (2.6) is asymptotically stable (with respect to $\mathbf{r}$, $\dot{\mathbf{r}}$, $\dot{\mathbf{s}}$).

Theorem 2.3. If $\mathbf{B} \equiv 0$ and the "reduced" potential V_p takes a stationary value at the point $\mathbf{r}_p^0$ which is not even a non-strict minimum of V_p, and this point is isolated from the other stationary points of the "reduced" potential, then the steady motion (2.6) is unstable.

Theorem 2.1 follows from (2.7), Theorem 2.2 follows from the asymptotic stability of the invariant set (2.2) and the Barbashin–Krasovskii theorem [17], and Theorem 2.3 follows from Krasovskii's theorem [17].

3. Now consider the case $\mathbf{B}(\mathbf{r}) \not\equiv 0$, assuming that $\mathbf{B}(\mathbf{r}_c^0) = 0$, where $\mathbf{r}_c^0$ is a solution of system (1.2). Then, if $\mathbf{p} = \mathbf{c}$, it follows from Eq. (2.7) that $\mathbf{r}_p^0 = \mathbf{r}_c^0$. We have thus proved the following theorem.

Theorem 3.1. If $\mathbf{B}(\mathbf{r}_c^0) = 0$ and $\mathbf{p} = \mathbf{c}$, then the steady motion (2.6) of a system subject to the action of dissipative forces—the derivatives of a Rayleigh function proportional to the kinetic energy—and constant forces applied only with respect to the cyclic variables, is identical with the steady motion of the corresponding conservative system.

In the case under consideration, the generalized energy equation has the form

$$\frac{d}{dt}(R_2 + V_p) = -2fR_2 - f(\mathbf{g}_p(\mathbf{r}), \dot{\mathbf{r}}) \tag{3.1}$$

The right-hand side of Eq. (3.1) is always sign-variable when $\mathbf{g}_p(\mathbf{r}) \neq 0$, and it is therefore impossible to investigate the stability of the steady motions (2.6) by direct application of the theorems of Lyapunov's direct method even when $\mathbf{p} = \mathbf{c}$ and $\mathbf{g}_p(\mathbf{r}_c^0) = 0$ $(\mathbf{B})(\mathbf{r}_c^0) = 0$. However, if $\mathbf{p} = \mathbf{c}$ and not only $\mathbf{g}_p(\mathbf{r})$ but also $\partial \mathbf{g}_p/\partial \mathbf{r}$ vanish at $\mathbf{r} = \mathbf{r}_c^0$, we can rewrite (3.1) as

$$d/dt(R_2(\mathbf{r}_c^0; \dot{\mathbf{r}}) + \delta^2 V_p(\mathbf{r})) = -2fR_2(\mathbf{r}_c^0; \dot{\mathbf{r}}) + o(\|\delta\mathbf{r}\|^2 + \|\dot{\mathbf{r}}\|^2) \tag{3.2}$$

$$\delta^2 V_p(\mathbf{r}) = \tfrac{1}{2}((\partial^2 V_p(\mathbf{r})/\partial \mathbf{r}^2)_0 \delta\mathbf{r}, \delta\mathbf{r}), \quad \delta\mathbf{r} = \mathbf{r} - \mathbf{r}_c^0$$

The subscript zero means that the relevant expression is evaluated at $\mathbf{r} = \mathbf{r}_c^0$.

As before, the right-hand side of (3.2) is of variable sign. However, it implies that if $\mathbf{p} = \mathbf{c}$, the linearized equations of the perturbed motion of the "reduced" system admit of an "energy" equation in the neighbourhood of its equilibrium position $\mathbf{r} = \mathbf{r}_p^0 = \mathbf{r}_c^0$, $\dot{\mathbf{r}} = 0$, namely

$$d/dt(R_2(\mathbf{r}_c^0;\ \dot{\mathbf{r}}) + \delta^2 V_p(\mathbf{r})) = -2fR_2(\mathbf{r}_c^0; \dot{\mathbf{r}}) \leqslant 0 \tag{3.3}$$

Thus, the following propositions are true.

Theorem 3.2. If $\mathbf{B}(\mathbf{r}_c^0) = 0$, $(\partial \mathbf{g}_p/\partial \mathbf{r})_0 = 0$, $\mathbf{p} = \mathbf{c}$ and all the eigenvalues of the matrix $(\partial^2 V_p/\partial \mathbf{r}^2)_0$ are positive, the steady motion (2.6) is asymptotically stable (with respect to $\mathbf{r}$, $\dot{\mathbf{r}}$ and $\dot{\mathbf{s}}$).

Theorem 3.3. If $\mathbf{B}(\mathbf{r}_c^0) = 0$, $(\partial \mathbf{g}_p/\partial \mathbf{r})_0 = 0$, $\mathbf{p} = \mathbf{c}$ and at least one of the eigenvalues of the matrix $(\partial^2 V_p/\partial \mathbf{r}^2)_0$ is negative, the steady motion (2.6) is unstable.

Indeed under the assumptions of Theorem 3.2, the trivial solution of the linearized equations of the perturbed motion of the "reduced" system is asymptotically stable in the neighbourhood of its position of equilibrium $\mathbf{r} = \mathbf{r}_p^0 = \mathbf{r}_c^0$, $\dot{\mathbf{r}} = 0$, by the Barbashin–Krasovskii theorem. Consequently, all the roots of the corresponding characteristic equation lie in the left half-plane. This means that this equilibrium of the "reduced" system is asymptotically stable (with respect to $\mathbf{r}$ and $\dot{\mathbf{r}}$) by virtue of the full equations of perturbed motion of the "reduced" system. Recalling that the "reduced" system describes the motion of the initial system in an asymptotically stable invariant set, we conclude that the steady motion (2.6) of the system is asymptotically stable (with respect to $\mathbf{r}$, $\dot{\mathbf{r}}$ and $\dot{\mathbf{s}}$).

Under the assumptions of Theorem 3.3, the characteristic equation of the linearized "reduced" system has a root in the right-hand half-plane. Consequently, the steady motion (2.6) is unstable, by Lyapunov's theorem on instability in the first approximation and a remark of Chetayev [5] (to prove instability, we can confine our attention to perturbed motions on the invariant set).

4. We will now consider the case in which $\mathbf{g}_p(\mathbf{r}) = \operatorname{grad} \Gamma_p(\mathbf{r})$, where $\Gamma_p(\mathbf{r})$ is some scalar function. The system may perform steady motions (2.6) if the constants $\mathbf{r}_p^0$ satisfy the following system of equations (see (2.7))

$$\partial W_{p,f}/\partial \mathbf{r} = \mathbf{0}; \quad W_{p,f} = V_p(\mathbf{r}) + f\Gamma_p(\mathbf{r}) \tag{4.1}$$

In addition, system (2.5) in this case admits of a generalized energy equation

$$d(R_2 + W_{p,f})/dt = -fR_2 \leqslant 0 \tag{4.2}$$

Let us call the function $W_{p,f}$ the perturbed "reduced" potential. If the dissipation constant f is sufficiently small, the solution of system (4.1) will be

$$\mathbf{r} = \mathbf{r}_p^0(f) = \mathbf{r}_p^0 + f\mathbf{r}_p^1 + \dots \tag{4.3}$$

where $\mathbf{r}_p^0$ is the solution of system (4.1) at $f = 0$, i.e. if $\mathbf{p} = \mathbf{c}$ is a solution of system (1.2). Under these conditions the steady motions (2.6) of the system have the form

$$\mathbf{r} = \mathbf{r}_p^0(f), \quad \dot{\mathbf{r}} = 0, \quad \mathbf{s} = \dot{\mathbf{s}}_p^0(f)t + \mathbf{s}^0, \quad \dot{\mathbf{s}} = \dot{\mathbf{s}}_p^0(f) = \mathbf{C}^{-1}(\mathbf{r}_p^0(f))\mathbf{p} \tag{4.4}$$

Obviously, the point $\mathbf{r}^0{}_p(f)$ gives the perturbed "reduced" potential $W_{p,f}$ a stationary value, and the steady motion (4.4) is asymptotically stable (unstable) if $\mathbf{r}_p^0(f)$ gives the perturbed "reduced" potential a strictly minimum value (does not give it even a non-strictly minimum value) and is isolated from other stationary points of the potential (see (4.1) and (4.2)).

Note that $\mathbf{g}_p(\mathbf{r}) = \operatorname{grad} \Gamma_p(\mathbf{r})$ if and only if the system is gyroscopically disconnected, i.e. if and only if $\mathbf{G}_p(\mathbf{r}) \equiv 0$ [18]. In particular, $\mathbf{g}_p(\mathbf{r}) = \operatorname{grad} \Gamma_p(\mathbf{r})$ if $\dim \mathbf{r} = 1$.

If $\mathbf{p} = \mathbf{c}$ and the coefficient f is sufficiently small, then $\mathbf{r}_p^0(f)$ is close to $\mathbf{r}_c^0$ (see (4.3)). In addition, if that is the case, the perturbed "reduced" potential $W_{p,f}(\mathbf{r})$ has a strict minimum (does not even have a non-strict minimum) at the point $\mathbf{r}_p^0(f)$ if all the eigenvalues of the matrix $\partial^2 V_c/\partial \mathbf{r}^2$ are positive (if the matrix $\partial^2 V_c/\partial \mathbf{r}^2$ has a negative eigenvalue) at the point $\mathbf{r}_c^0$. We have thus proved the following propositions.

Theorem 4.1. The steady motions (4.4) of a gyroscopically disconnected system subject to dissipative forces—the derivatives of a Rayleigh function proportional to the kinetic energy—and constant forces applied only with respect to the cyclic variables, are close at $\mathbf{p} = \mathbf{c}$ to the steady motions (1.1) of the corresponding conservative system.

Theorem 4.2. The steady motion (4.4) is asymptotically stable (with respect to $\mathbf{r}$, $\dot{\mathbf{r}}$ and $\dot{\mathbf{s}}$) if $\mathbf{p} = \mathbf{c}$, the dissipation coefficient is small and the reduced potential of the corresponding conservative system has a strict minimum in the corresponding steady motion (1.1), where the existence of the minimum is already determined by the second variation of the reduced potential.

Theorem 4.3. The steady motion (4.4) is unstable if $\mathbf{p} = \mathbf{c}$, the dissipation coefficient is small and the reduced potential of the corresponding conservative system does not have even a non-strict minimum at the corresponding steady motion (1.1), the absence of the minimum being determined by the second variation of the reduced potential.

Theorem 4.1 follows from (4.3); Theorems 4.2 and 4.3 follow from relation (4.2) and from the fact that (when $\mathbf{p} = \mathbf{c}$ and f is small) the perturbed "reduced" potential $W_{p,f}(\mathbf{r})$ is close to the reduced potential $V_c(\mathbf{r})$.

Remark. Theorems 4.1–4.3 remain valid not only when $\mathbf{p} = \mathbf{c}$ but also when $\mathbf{p}$ is close to $\mathbf{c}$. Theorem 4.1 also holds for gyroscopically connected systems (see (2.7)).

Thus, even in the simple case just considered, when the Rayleigh dissipation function is proportional to the kinetic energy, the steady motions of a system subject to dissipative and constant forces applied only with respect to the cyclic variables, and the conditions for their stability, are identical with the steady motions of the corresponding conservative system and the conditions for its stability, or are close to the latter under additional conditions as indicated above. In the general case no such conclusions are possible (see the next section).

5. Consider a physical pendulum suspended on a horizontal axis Ox, which may revolve around the vertical OZ. Let us assume that the Ox axis is one of the principal axes of inertia of the body about O. Denote the two other principal axes by Oy and Oz and assume that the centre of mass of the body lies on Oz at a distance d from O. Let m be the mass of the body, let A, B and C be its principal moments of inertia about the Ox, Oy and Oz, axes and g is the acceleration due to gravity. The position of the body is defined by the coordinates θ and ψ, where θ is the angle between the downward vertical and the Oy axis, and ψ is the angle of rotation of the axis Ox about OZ. Then the kinetic energy T and the potential energy V of the body are

$$T = \tfrac{1}{2}[A\dot{\theta}^2 + J(\theta)\dot{\psi}^2], \quad V = -mgd\cos\theta; \quad J(\theta) = B\sin^2\theta + C\cos^2\theta$$

Let us assume that the system is subject to dissipative forces (derivatives of the Rayleigh function fT) and a constant torque fp applied along the OZ axis. Then the equations of motion of the body are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} = \frac{\partial T}{\partial \theta} - \frac{\partial V}{\partial \theta} - f\frac{\partial T}{\partial \dot{\theta}}, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\psi}} = fp - f\frac{\partial T}{\partial \dot{\psi}} \tag{5.1}$$

In accordance with our previous results, we define the Routh function $R = T - V - p\psi$, where the generalized cyclic velocity $\dot\psi$ is eliminated using the particular integral $\partial T/\partial\dot\psi = p$ of system (5.1). Using these conditions

$$R = \tfrac{1}{2}A\dot\theta^2 - V_p(\theta), \quad V_p = V + \tfrac{1}{2}p^2 / J(\theta) \tag{5.2}$$

Obviously, these relations are identical with those defining the Routh function and the reduced potential of the corresponding conservative system, which is obtained from the system under consideration by putting $f = 0$ (p is then the constant of the general integral $\partial T/\partial\dot\psi = p$).

Taking into account that the system is gyroscopically disconnected (the kinetic energy does not contain $\dot\theta\dot\psi$), we conclude from the results of Section 2 that all our conclusions concerning the existence and stability of steady motions of the physical pendulum in the classical setting ($f = 0$) extend to the case $f > 0$.

Thus, in this problem steady motions

$$\theta = 0, \ \dot\theta = 0, \ \dot\psi = p/C \tag{5.3}$$

$$\theta = \pi, \ \dot\theta = 0, \ \dot\psi = p/C \tag{5.4}$$

$$\theta = \theta_p, \ \dot\theta = 0, \ \dot\psi = p/J(\theta_p) \tag{5.5}$$

exist where θ_p is a root of the equation

$$\frac{p^2}{mdg} = \frac{J^2(\theta)}{(B-C)\cos\theta} \tag{5.6}$$

The stability (instability) conditions for the steady motions (5.3)–(5.5) are, respectively

$$mgd - \frac{(B-C)p^2}{C^2} > 0 \ \ (<0) \tag{5.7}$$

$$-mgd - \frac{(B-C)p^2}{C^2} > 0 \ \ (<0) \tag{5.8}$$

$$(B-C)\left[1 + \frac{4(B-C)\cos^2\theta_p}{J(\theta)_p}\right] > 0 \ \ (<0) \tag{5.9}$$

and they are identical with the stability (instability) conditions for the system without dissipative and constant forces (see, for example [14]).

We will now consider the case in which the Rayleigh dissipation function $f\Phi$ has the same structure as the kinetic energy of the body but is not proportional to it

$$\Phi = \tfrac{1}{2}[D\dot\theta^2 + I(\theta)\dot\psi^2], \quad I(\theta) = E\sin^2\theta + F\cos^2\theta \ \ (D > 0, \ E > 0, \ F > 0)$$

Then the equations of motion of the body differ from (5.1) in that $f\partial T/\partial\theta$ on the right of the equations must be replaced by $f\partial T\Phi/\partial\theta$ and $f\partial T/\partial\dot\psi$ by $f\partial\Phi/\partial\dot\psi$; these equations need not necessarily admit of even particular integrals.

Nevertheless, they do have steady solutions

$$\theta = 0, \ \dot\theta = 0, \ \dot\psi = p/F \tag{5.10}$$

$$\theta = \pi, \ \dot\theta = 0, \ \dot\psi = p/F \tag{5.11}$$

$$\theta = \theta_p, \dot\theta = 0, \ \dot\psi = p/I(\theta_p) \tag{5.12}$$

where θ_p is a root of the equation

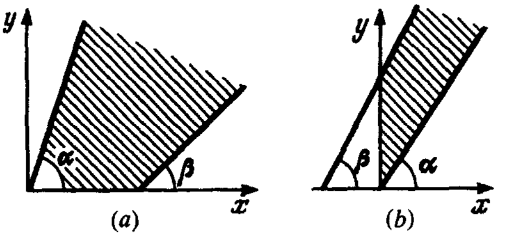

Fig. 1.

$$\frac{p^2}{mdg}=\frac{I^2(\theta)}{(B-C)\cos\theta} \tag{5.13}$$

In a certain sense, solutions (5.10)–(5.12) are analogous to (5.3)–(5.5) (even when allowance is made for the difference between Eqs (5.6) and (5.13)).

Stability investigation of the steady motions (5.10)–(5.12), based on analysing the roots of the characteristic equation of the linearized system of equations of the perturbed motion, implies the following results. The solutions (5.10) and (5.11) are stable (unstable) provided that, respectively

$$mgd-(p/F)^2(B-C)>0 \quad (<0) \tag{5.14}$$

$$mgd+(p/F)^2(B-C)<0 \quad (>0) \tag{5.15}$$

Investigating the stability of the motion (5.12), let us assume that f is a small quantity. Then the stability (instability) conditions are

$$\begin{aligned}&(B-C)[D(B\sin^2\theta_p+C\cos^2\theta_p)(B+3(B-C)\cos^2\theta_p)+\\&+4A\cos^2\theta_p(FB-EC)]>0 \quad (<0)\\&(B-C)[E+3(E-F)\cos^2\theta_p]>0 \quad (<0)\end{aligned} \tag{5.16}$$

A geometric interpretation of these conditions is shown in Fig. 1, where we have introduced the notation $x = F/B$, $y = E/B$. The angle α is determined from the equation $\operatorname{tg}\alpha = 1 + (3\cos^2\theta)^{-1}$, and β from the equation $\operatorname{tg}\beta = C/B$. The stability domain is shown hatched. Cases a and b correspond to the conditions $C < B$ and $3C\cos^2\theta > 3B\cos^2\theta + B$, respectively. If these conditions do not hold, the motions (5.12) are unstable.

Thus, if the Rayleigh dissipation function is not proportional to the kinetic energy, the influence of dissipative and constant forces may destabilize motions of the system that are s table when there are no such forces.

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